Limiti di funzioni

Sia $f : A \subset \mathbb{R} \to \mathbb{R}$, sia $x_0 \in \overline{\mathbb{R}}$ di accumulazione per A, sia $l \in \mathbb{R}$

$$\lim_{x \to x_0} f(x) = l \quad \Leftrightarrow \quad \forall \epsilon > 0, \exists \delta > 0 : \forall x \in A, 0 < |x - x_0| < \delta \Rightarrow |f(x) - l| < \epsilon$$

$$\lim_{x \to x_0^-} f(x) = l \quad \Leftrightarrow \quad \forall \epsilon > 0, \exists \delta > 0 : \forall x \in A, x_0 - \delta < x < x_0 \Rightarrow |f(x) - l| < \epsilon$$

$$\lim_{x \to x_0^+} f(x) = l \quad \Leftrightarrow \quad \forall \epsilon > 0, \exists \delta > 0 : \forall x \in A, x_0 < x < x_0 + \delta \Rightarrow |f(x) - l| < \epsilon$$

Verificare i seguenti limiti:

1. $\lim_{x \to 2} \log_2 x = 1$
2. $\lim_{x \to 0} \sin^2 \frac{1}{x} + \cos^2 \frac{1}{x} = 1$
3. $\lim_{x \to 0} x^2 \sin^2 \frac{1}{x} = 0$
4. $\lim_{x \to 0^+} \sqrt{x} \sin 3^{\frac{1}{x}} = 0$
5. $\lim_{x \to 0} \sin^2 \frac{1}{x} \nexists$

Limiti infiniti e all'infinito

Sia $f : A \subset \mathbb{R} \to \mathbb{R}$, sia $x_0 \in \mathbb{R}$ di accumulazione per A, allora:

$$\lim_{x \to x_0} f(x) = +\infty \quad \Leftrightarrow \forall M > 0, \exists \delta > 0 : \forall x \in A, 0 < |x - x_0| < \delta \Rightarrow f(x) > M$$

$$\lim_{x \to x_0} f(x) = -\infty \quad \Leftrightarrow \forall M > 0, \exists \delta > 0 : \forall x \in A, 0 < |x - x_0| < \delta \Rightarrow f(x) < -M$$

Sia A illimitato superiormente e/o inferiormente:

$$\lim_{x \to \pm\infty} f(x) = l \quad \Leftrightarrow \forall \epsilon > 0, \exists N > 0 : \forall x \in A, |x| > N \Rightarrow |f(x) - l| < \epsilon$$

$$\lim_{x \to +\infty} f(x) = +\infty \quad \Leftrightarrow \forall M > 0, \exists N > 0 : \forall x \in A, x > N \Rightarrow f(x) > M$$

$$\lim_{x \to -\infty} f(x) = +\infty \quad \Leftrightarrow \forall M > 0, \exists N > 0 : \forall x \in A, x < -N \Rightarrow f(x) > M$$

$$\lim_{x \to +\infty} f(x) = -\infty \quad \Leftrightarrow \forall M > 0, \exists N > 0 : \forall x \in A, x > N \Rightarrow f(x) < -M$$

$$\lim_{x \to -\infty} f(x) = -\infty \quad \Leftrightarrow \forall M > 0, \exists N > 0 : \forall x \in A, x < -N \Rightarrow f(x) < -M$$

Verificare i seguenti limiti

6. $\lim_{x \to +\infty} \arctan x = \frac{\pi}{2}$
7. $\lim_{x \to -1^+} \log_3(x+1) = -\infty$
8. $\lim_{x \to -\infty} 4^{-x} = +\infty$
9. $\lim_{x \to +\infty} \sqrt{x^2+4x+3} + x = +\infty$

Confronto locale

Siano $f : I_{x_0} \setminus \{x_0\} \to \mathbb{R}$, $g : I_{x_0} \setminus \{x_0\} \to \mathbb{R} \setminus \{0\}$ entrambe infinitesime (infinite) in $x_0 \in \overline{\mathbb{R}}$:

- f e g sono infinitesimi (infiniti) dello stesso ordine per $x \to x_0$:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = l \in \mathbb{R} \setminus \{0\}$$

- f e g sono infinitesimi (infiniti) equivalenti per $x \to x_0$:

$$f \sim g \quad \text{per} \quad x \to x_0 \quad \Leftrightarrow \quad \lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$$

- f è infinitesimo di ordine superiore a g per $x \to x_0$:

$$f = o(g) \quad \text{per} \quad x \to x_0 \quad \Leftrightarrow \quad \lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$$

- f è infinito di ordine superiore a g per $x \to x_0$: $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \infty$

Limiti notevoli

- $\lim_{x\to 0} \frac{\sin x}{x} = 1$
- $\lim_{x\to 0} \frac{\arctan x}{x} = 1$
- $\lim_{x\to 0} \frac{\arcsin x}{x} = 1$
- $\lim_{x\to 0} \frac{e^x-1}{x} = 1$
- $\lim_{x\to 0} \frac{\log(x+1)}{x} = 1$
- $\lim_{x\to +\infty} \left(1 + \frac{1}{x}\right)^x = e$
- $\lim_{x\to +\infty} \frac{\log^\beta x}{x^\alpha} = 0, \qquad \forall \alpha > 0, \beta \in \mathbb{R}$
- $\lim_{x\to +\infty} \frac{x^\beta}{\alpha^x} = 0, \qquad \forall \alpha > 0, \beta \in \mathbb{R}$
- $\lim_{x\to +\infty} \frac{\alpha^x}{x^x} = 0, \qquad \forall \alpha > 0$

Limiti notevoli

Calcolare i seguenti limiti notevoli

10. $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$
11. $\lim_{x\to 0} \frac{\tan x}{x} = 1$
12. $\lim_{x\to 0^+} x^\alpha (\log_a x)^\beta = 0, \qquad \alpha > 0, \beta \in \mathbb{R}$
13. $\lim_{x\to 0} \frac{a^x-1}{x} = \log a$
14. $\lim_{x\to 0^+} x^x = 1$
15. $\lim_{x\to +\infty} x^{\frac{1}{x}} = 1$
16. $\lim_{x\to \pm\infty} \left(1 + \frac{a}{x}\right)^x = e^a$
17. $\lim_{x\to 0} (1 + ax)^{\frac{1}{x}} = e^a$

Esercizi

18. $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = 1$
19. $\lim_{x\to \frac{\pi}{6}} \frac{1+2\cos\left(\frac{\pi}{2}+x\right)}{1-4\sin^2(\pi+x)} = \frac{1}{2}$
20. $\lim_{x\to 1} \frac{\sqrt{x}-1}{x^2-1} = \frac{1}{4}$
21. $\lim_{x\to 0} \frac{\sin^2 x}{x} = 0$
22. $\lim_{x\to 0} \frac{\sin \alpha x}{\sin \beta x} = \frac{\alpha}{\beta}$
23. $\lim_{x\to +\infty} \log_4 (1 + 2 \cdot 4^x) - x = \frac{1}{2}$
24. $\lim_{x\to -\infty} \sqrt{x^2 + 4x + 3} + x = 2$
25. $\lim_{x\to +\infty} \frac{\left(\frac{x+2}{x+1}\right)^{2x}}{\log\left[\sin \frac{3}{\pi}\right]-\log \frac{1}{x}} = \frac{e^2}{\log 3}$

Esercizi

26. $\lim_{x\to 0} \frac{5^x-7^x}{x} = \log \frac{5}{7}$
27. $\lim_{x\to +\infty} \frac{5^x-7^x}{x} = -\infty$
28. $\lim_{x\to \frac{\pi}{2}} \left(\sin^2 x\right)^{\tan^2 x} = e^{-1}$
29. $\lim_{x\to 0^+} \frac{\left(\sqrt{e}\right)^{\sin x}-1}{\left[\log(1+\sqrt{x})\right]^2} = \frac{1}{2}$
30. $\lim_{x\to 0} (1+\sin x)^{\frac{1}{x}} = e$
31. $\lim_{x\to 0^+} \frac{1-\sin x-\cos^2 x}{e^{x^2}-1} = -\infty$
32. $\lim_{x\to 0^+} \frac{(1+2x)^{\log x}-1}{\sin x \log x^3} = \frac{2}{3}$
33. $\lim_{x\to +\infty} \left[(\log\log x)^{\log x} - x\,(\log x)^{\log\log x}\right] = -\infty$
34. $\lim_{x\to 0^+} [\log(e-x)]^{\frac{1}{\sqrt{x}}} = 1$